A note about algebras obtained by the Cayley-Dickson process

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Abstract. In this paper, we generalize the concepts of level and sublevels of a composition algebra to algebras obtained by the Cayley-Dickson process. In 1967, R. B. Brown constructed, for every $t \in \mathbb{N}$, a division algebra A_t of dimension 2^t over the power-series field $K\{X_1, X_2, ..., X_t\}$. This gives us the possibility to construct a division algebra of dimension 2^t and prescribed level 2^k, k, $t \in \mathbb{N}^*$.

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1. Introduction

First, we recall some definitions and properties for nonassociative algebras.

Let V be a vector space over a field K and $b : V \times V \to K$ be a bilinear form. The pair (V, b) is called a *symmetric bilinear space* over the field K.

Let A be an algebra of dimension n over a field K and let $f_1, ..., f_n$ be a basis for A over K. The multiplication in algebra A is given by the relations $f_i f_j = \sum\limits_{k=1}^{n} \alpha_{ijk} f_k$, where $\alpha_{ijk} \in K$ and $i, j = 1, ...n$. If $K \subset F$ is a field extension, the algebra $A_F = F \otimes_K A$ is called the *scalar extension* of A to an algebra over F. The elements of A_F are the forms $\sum\limits_{i=1}^{n} \alpha_i \otimes f_i$ and we denote them $\sum\limits_{i=1}^{n} \alpha_i f_i$, $\alpha_i \in F$.

An algebra A over a field K is called *quadratic* if A is a unitary algebra and, for all $x \in A$, there are $a, b \in K$ such that $x^2 = ax + b1$, $a, b \in K$. The subset $A_0 = \{x \in A - K \mid x^2 \in K1\}$ is a linear subspace of A and $A = K \cdot 1 \oplus A_0$. In this paper, we assume that $char K \neq 2$. This decomposition allows us to define a linear form $t : A \to K$, a linear map $i : A \to A_0$ such that $x = t(x) \cdot 1 + i(x), \forall x \in A$, a symmetric bilinear form, $(,) : A \times A \to K$, $(x, y) = -\frac{1}{2}t(xy + yx)$ and a quadratic form $n : A \to K$, $n(x) = (t(x))^2 + (i(x), i(x))$. The element $\bar{x} = t(x) \cdot 1 - i(x)$ is called the *conjugate* of x. The quadratic form n is called *anisotropic* if $n(x) = 0$ implies $x = 0$. In this case, the algebra A is called also *anisotropic,* otherwise A is *isotropic.*

We can decompose the algebra A under the form $A = Sym(A) \oplus Skew(A)$, where $Sym(A) = \{x \in A \mid x = \bar{x}\}$, $Skew(A) = \{x \in A \mid x = -\bar{x}\}$.

Definition 1.1. i) Let $b_1 : V_1 \times V_1 \to K$ and $b_2 : V_2 \times V_2 \to K$ be two bilinear forms. Then the vector space $V_1 \oplus V_2$ with the bilinear form denoted $b_1 \perp b_2$ and defined by

$$(b_1 \perp b_2)((x_1, x_2), (y_1, y_2)) = b_1(x_1, y_1) + b_2(x_2, y_2),$$

is called the *external orthogonal sum of the vector spaces* (V_1, b_1) and (V_2, b_2) and it is denoted by $(V_1, b_1) \perp (V_2, b_2)$.

ii) Let $\varphi : V \to K$ be a quadratic form over V and $t \in \mathbb{N}^*$. The quadratic form $\underbrace{\varphi \perp ... \perp \varphi}_{t \ times}$ is denoted by $t \times \varphi$.[La, Ma; 01]

A bilinear space (V, b) *represents* $\alpha \in K$ if there is an element $x \in V, x \neq 0$, with $b(x, x) = \alpha$. The space is called *universal* if (V, b) represents all $\alpha \in K$. Every isotropic bilinear space V, $V \neq \{0\}$, is universal.[Sch; 85, Lemma 4.11., p. 14]

Every symmetric matrix is congruent with a diagonal matrix. For $\alpha_1, ..., \alpha_m$ in the field K, we define *the bilinear space* $< \alpha_1, ..., \alpha_m >$ as the space with the quadratic form given by the matrix

$$A = \begin{pmatrix} \alpha_1 & & 0 \\ & ... & \\ 0 & & \alpha_m \end{pmatrix}$$

and it is denoted $< \alpha_1, ..., \alpha_m > := < A >$. Since every symmetric bilinear space is the orthogonal sum of one-dimensional subspaces, then every symmetric bilinear space is isomorphic to a space $< \alpha_1, ..., \alpha_m >$.

Definition 1.2. A quadratic form ψ is a *subform* of the form φ if $\varphi \simeq \psi \perp \phi$, for some quadratic form ϕ. We denote $\psi < \varphi$.

Definition 1.3. The two dimensional symmetric bilinear vector space $H =< 1, -1 >$ is called *hyperbolic plane*. A quadratic form φ is *hyperbolic* if $\varphi \simeq H \perp ... \perp H$.

Definition 1.4. i) Let $A = (a_{ij}) \in M_m(K)$ and $B = (b_{ij}) \in M_n(K)$. We denote by $A \otimes B$ the matrix $C \in M_{mn}(K)$, of the form

$$C = A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & ... & a_{1m}B \\ ... & ... & ... & ... \\ a_{m1}B & a_{m2}B & ... & a_{mm}B \end{pmatrix}.$$

The matrix $A \otimes B$ is called the *tensor product* of A and B.

ii) The *tensor product of two bilinear spaces* $< A >$ and $< B >$ is the bilinear space defined by:

$$< A >< B >=< A > \otimes < B >:=< A \otimes B > .$$

iii) If $< A >$ and $< B >$ are two symmetric bilinear spaces with the quadratic forms φ and ψ, then the quadratic form of the space $< A >< B >$ is denoted $\varphi \otimes \psi$ and it is called the *orthogonal product* of the quadratic forms φ and ψ.

For $x \in K^*$ the scalar multiple $< x > \otimes \varphi$ is denoted $x\varphi$.[La, Ma; 01]

Proposition 1.5. *Let φ_1, φ_2 be two quadratic forms over K. Let $K(x)$ be the rational function field in one variable over K. Then the quadratic form $\varphi_1 \perp x\varphi_2$ is isotropic over $K(x)$ if and only if φ_1 or φ_2 are isotropic over K.* [La, Ma;01, p.1823, Theorem 1.1.]

Definition 1.6. Let φ be a regular(nondegenerate) $n-$dimensional quadratic form over K, $n \in N, n > 1$, which is not isometric to the hyperbolic plane. We may consider φ as a homogeneous polynomial of degree 2, $\varphi(X) = \varphi(X_1, ...X_n) = \sum a_{ij}X_iX_j$, $a_{ij} \in K^*$. The *functions field of* φ, denoted $K(\varphi)$, is the quotient field of the integral domain

$$K[X_1, ..., X_n] \, / \, (\varphi(X_1, ..., X_n)) \, .$$

Since $(X_1, ..., X_n)$ is a non-trivial zero, φ is isotropic over $K(\varphi)$. (We remark that $\varphi(X)$ is irreducible).[Sch;85]

Definition 1.7. The field K is a *formally real field* if -1 is not a sum of squares in K.

Each formally real field has characteristic zero.

Definition 1.8. For $n \in \mathbb{N}^*$ a *$n-$fold Pfister form* over the field K is a quadratic form which has the expression:

$$< 1, a_1 > \otimes ... \otimes < 1, a_n >, \ a_1, ..., a_n \in K^*.$$

Remark 1.9. A Pfister form φ has the expression

$$< 1, a_1 > \otimes ... \otimes < 1, a_n > = < 1, a_1, a_2, ..., a_n, a_1 a_2, ..., a_1 a_2 a_3, ..., a_1 a_2 ... a_n > .$$

If $\varphi =< 1 > \perp \varphi'$, then φ' is called *the pure subform* of φ. A Pfister form is hyperbolic if and only if is isotropic. It results that a Pfister form is isotropic if and only if its pure subform is isotropic. [Sch; 85]

A *composition algebra* is an algebra A with two mappings

$$(,) : A \times A \to K, \, n : A \to K,$$

the *bilinear form* and the corresponding *norm form*, such that $n(xy) = n(x) n(y), \forall x, y \in A$. A unitary composition algebra is called a *Hurwitz algebra.* Hurwitz algebras have dimensions $1, 2, 4, 8$.

Since over fields, the classical Cayley-Dickson process generates all possible Hurwitz algebras, in the following, we recall shortly the *Cayley-Dickson process.*

Let A be a finite dimensional unitary algebra over a field K with $charK \neq 2$ and an involution $\phi : A \to A, \phi(a) = \overline{a},$ where $a + \overline{a}$ and $a\overline{a} \in K \cdot 1, \forall a \in A$. Since A is unitary, we identify K with $K \cdot 1$ and we consider $K \subseteq A$.

Let $\alpha \in K$ be a fixed non-zero element. We define the following algebra multiplication on the vector space $A \oplus A$.

$$(a_1, a_2)(b_1, b_2) = \left(a_1 b_1 + \alpha \overline{b_2} a_2, a_2 \overline{b_1} + b_2 a_1\right). \qquad (1.1.)$$

We obtain an algebra structure over $A \oplus A$. This algebra, denoted by (A, α), is called the *algebra obtained from A by the Cayley-Dickson process(or the associated doubled algebra of A).* A is canonically isomorphic with a subalgebra of the algebra (A, α) (in fact, we denote $(1, 0)$ by 1 and this is the

identity in (A,α)) and $\dim(A,\alpha) = 2\dim A$. Taking $u = (0,1) \in A \oplus A$, $u^2 = \alpha \cdot 1$ and $(A,\alpha) = A \oplus Au$.

Let $x = a_1 \cdot 1 + a_2 u \in (A,\alpha)$. The element $\overline{x} = \bar{a}_1 - a_2 u$ is called *the conjugate of the element* x. We remark that $x + \overline{x} = a_1 + \overline{a_1} \in K \cdot 1$ and $x\overline{x} = a_1\overline{a_1} + \alpha a_2 \overline{a_2} \in K \cdot 1$. The map ψ

$$\psi : (A,\alpha) \to (A,\alpha)\ , \quad \psi(x) = \bar{x}\,,$$

is an involution of the algebra (A,α), extending the involution ϕ. If x, $y \in (A,\alpha)$, it follows that $\overline{xy} = \overline{y}\,\overline{x}$.

For $x \in A$, we denote $t(x) \cdot 1 = x + \overline{x} \in K$, $n(x) \cdot 1 = x\overline{x} \in K$, and these are called the *trace*, respectively, the *norm* of the element $x \in A$. If $z \in (A,\alpha)$, $z = x + yu$, then $z + \overline{z} = t(z) \cdot 1$ and $z\overline{z} = \overline{z}z = n(z) \cdot 1$, where $t(z) = t(x)$ and $n(z) = n(x) - \alpha n(y)$. It follows that $(z + \overline{z})\, z = z^2 + \overline{z}z = z^2 + n(z) \cdot 1$ and

$$z^2 - t(z)\, z + n(z) = 0, \forall z \in (A,\alpha)\,,$$

therefore each algebra obtained by the Cayley-Dickson process is quadratic. All algebras A obtained by the Cayley-Dickson process are *flexible* (i.e. $x(yx) = (xy)\,x, \forall x, y \in A$) and *power-associative* (i.e. for each $a \in A$, the subalgebra of A generated by a is associative). Moreover, the following conditions are fulfilled:

$$t(xy) = t(yx)\,, t((xy)\,z) = t(x\,(yz))\,, \forall x, y, z \in (A,\alpha)\,. \qquad (1.2.)$$

Remark 1.10. If we take $A = K$ and apply this process t times, $t \geq 1$, we obtain an algebra over K, $A_t = K\{\alpha_1, ..., \alpha_t\}$. By induction, in this algebra we find a basis $\{1, f_2, ..., f_q\}, q = 2^t$, satisfying the properties:

$$f_i^2 = \alpha_i 1,\ \alpha_i \in K, \alpha_i \neq 0,\ i = 2, ..., q.$$

$$f_i f_j = -f_j f_i = \beta_k f_k,\ \beta_k \in K,\ \beta_k \neq 0, i \neq j, i, j = 2, ...q,$$

β_k and f_k being uniquely determined by f_i and f_j.

As an example, we consider the octonion generalized algebra $O(\alpha, \beta, \gamma)$, with basis $\{1, f_2, ..., f_8\}$, having the multiplication table:

$\cdot$	1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
1	1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
f_2	f_2	α	f_4	$-\alpha f_3$	f_6	$-\alpha f_5$	$-f_8$	αf_7
f_3	f_3	$-f_4$	β	βf_2	f_7	f_8	$-\beta f_5$	$-\beta f_6$
f_4	f_4	αf_3	$-\beta f_2$	$-\alpha\beta$	f_8	$-\alpha f_7$	βf_6	$-\alpha\beta f_5$
f_5	f_5	$-f_6$	$-f_7$	$-f_8$	γ	γf_2	γf_3	γf_4
f_6	f_6	αf_5	$-f_8$	αf_7	$-\gamma f_2$	$-\alpha\gamma$	$-\gamma f_4$	$\alpha\gamma f_3$
f_7	f_7	f_8	βf_5	$-\beta f_6$	$-\gamma f_3$	γf_4	$-\beta\gamma$	$-\beta\gamma f_2$
f_8	f_8	$-\alpha f_7$	βf_6	$\alpha\beta f_5$	$-\gamma f_4$	$-\alpha\gamma f_3$	$\beta\gamma f_2$	$\alpha\beta\gamma$

If $x \in A_t, x = x_1 1 + \sum\limits_{i=2}^{q} x_i f_i$, then $\bar{x} = x_1 1 - \sum\limits_{i=2}^{q} x_i f_i$ and $t(x) = 2x_1, n(x) = x_1^2 - \sum\limits_{i=2}^{q} \alpha_i x_i^2$. In the above decomposition of x, we call x_1 the *scalar part* of x and $x'' = \sum\limits_{i=2}^{q} x_i f_i$ the *pure part* of x. If we compute $x^2 = x_1^2 + x''^2 + 2x_1 x'' = x_1^2 + \alpha_1 x_2^2 + \alpha_2 x_3^2 - \alpha_1\alpha_2 x_4^2 + \alpha_3 x_5^2 - ... - (-1)^t (\prod\limits_{i=1}^{t} \alpha_i) x_q^2 + 2x_1 x''$, the scalar part of x^2 is represented by the quadratic form

$$T_C =< 1, \alpha_1, \alpha_2, -\alpha_1\alpha_2, \alpha_3, ..., (-1)^t (\prod_{i=1}^{t} \alpha_i) >=< 1, \beta_2, ..., \beta_q > \qquad (1.3.)$$

and, since $x''^2 = \alpha_1 x_2^2 + \alpha_2 x_3^2 - \alpha_1\alpha_2 x_4^2 + \alpha_3 x_5^2 - ... - (-1)^t (\prod\limits_{i=1}^{t} \alpha_i) x_q^2 \in K$, it is represented by the quadratic form

$$T_P =< \alpha_1, \alpha_2, -\alpha_1\alpha_2, \alpha_3, ..., (-1)^t (\prod_{i=1}^{t} \alpha_i) >=< \beta_2, ..., \beta_q > . \qquad (1.4.)$$

The quadratic form T_C is called *the trace form*, and T_P *the pure trace form* of the algebra A_t. We remark that $T_C =< 1 > \perp T_P$, and the norm $n = n_C =< 1 > \perp -T_P$, resulting that
$n_C =< 1, -\alpha_1, -\alpha_2, \alpha_1\alpha_2, \alpha_3, ..., (-1)^{t+1} (\prod\limits_{i=1}^{t} \alpha_i) >=< 1, -\beta_2, ..., -\beta_q > .$

Using Remark 1.9., the trace form n_C has the form $n_C = < 1, -\alpha_1 > \otimes ... \otimes < 1, -\alpha_t >$ and it is a Pfister form.

Using the above notation, we have that $x^2 = t(x)\, x - n(x)\, 1 = -n(x)\, 1 + 2x_1(x_1 + x'') = 2x_1^2 - n(x) + 2x_1 x''$. It results that $T_C(x) = 2x_1^2 - n(x)$, then

$T_C(x) = \dfrac{(t(x))^2}{2} - n_C(x)$. But $(t(x))^2 = t(x^2) + 2n_C(x)$, then

$T_C(x) = \dfrac{t(x^2)}{2}$.

2. Brown's construction of division algebras

In 1967, R. B. Brown constructed, for every t, a division algebra A_t of dimension 2^t over the power-series field $K\{X_1, X_2, ..., X_t\}$. We present shortly this construction, using, instead of power-series fields over K (as it was considered by R.B. Brown), the polynomial rings over K and their fields of fractions(the rational functions field).

First of all, we remark that if an algebra A is finite-dimensional, then it is a division algebra if and only if A does not contain zero divisors ([Sc;66]). For every t we construct a division algebra A_t over a field F_t. Let $X_1, X_2, ..., X_t$ be t algebraically independent indeterminates over the field K and $F_t = K(X_1, X_2, ..., X_t)$ be the rational functions field. For $i = 1, ..., t$, we construct the algebra A_i over the rational functions field $K(X_1, X_2, ..., X_i)$ by setting $\alpha_j = X_j$ for $j = 1, 2, ..., i$. Let $A_0 = K$. By induction over i, assuming that A_{i-1} is a division algebra over the field $F_{i-1} = K(X_1, X_2, ..., X_{i-1})$, we may prove that the algebra A_i is a division algebra over the field $F_i = K(X_1, X_2, ..., X_i)$.

Let $A_{F_i}^{i-1} = F_i \otimes_{F_{i-1}} A_{i-1}$. For $\alpha_i = X_i$ we apply the Cayley-Dickson process to algebra $A_{F_i}^{i-1}$. The obtained algebra, denoted A_i, is an algebra over the field F_i and has dimension 2^i.

Let

$$x = a + bv_i,\ y = c + dv_i,$$

be nonzero elements in A_i such that $xy = 0$, where $v_i^2 = \alpha_i$. Since

$$xy = ac + X_i\bar{d}b + (b\bar{c} + da)\,v_i = 0,$$

we obtain

$$ac + X_i\bar{d}b = 0 \qquad (1.5.)$$

and

$$b\bar{c} + da = 0. \qquad (1.6.)$$

But, the elements $a, b, c, d \in A_{F_i}^{i-1}$ are different from zero. Indeed, we have:

i) If $a = 0$ and $b \neq 0$, then $c = d = 0 \Rightarrow y = 0$, false;

ii) If $b = 0$ and $a \neq 0$, then $d = c = 0 \Rightarrow y = 0$, false;

iii) If $c = 0$ and $d \neq 0$, then $a = b = 0 \Rightarrow x = 0$, false;

iv) If $d = 0$ and $c \neq 0$, then $a = b = 0 \Rightarrow x = 0$, false.

It results $b \neq 0, a \neq 0, d \neq 0, c \neq 0$. If $\{1, f_2, ..., f_{2^{i-1}}\}$ is a basis in A_{i-1}, then $a = \sum_{j=1}^{2^{i-1}} g_j(1 \otimes f_j) = \sum_{j=1}^{2^{i-1}} g_j f_j, g_j \in F_i, g_j = \frac{g_j'}{g_j''}, g_j', g_j'' \in K[X_1, ..., X_i]$, $g_j'' \neq 0$, $j = 1, 2, ...2^{i-1}$, where $K[X_1, ..., X_t]$ is the polynomial ring. Let a_2 be the less common multiple of $g_1'', ...g_{2^{i-1}}''$, then we can write

$a = \frac{a_1}{a_2}$, where $a_1 \in A_{F_i}^{i-1}, a_1 \neq 0$. Analogously, $b = \frac{b_1}{b_2}, c = \frac{c_1}{c_2}, d =$

$\frac{d_1}{d_2}, b_1, c_1, d_1 \in A_{F_i}^{i-1} - \{0\}$ and $a_2, b_2, c_2, d_2 \in K[X_1, ..., X_t] - \{0\}$.

If we replace in the relations (1.5) and (1.6), we obtain

$$a_1 c_1 d_2 b_2 + X_i \bar{d}_1 b_1 a_2 c_2 = 0 \quad (1.7.)$$

and

$$b_1 \bar{c}_1 d_2 a_2 + d_1 a_1 b_2 c_2 = 0. \quad (1.8.)$$

If we denote $a_3 = a_1 b_2, b_3 = b_1 a_2, c_3 = c_1 d_2, d_3 = d_1 c_2$, $a_3, b_3, c_3, d_3 \in A_{F_i}^{i-1} - \{0\}$, the relations (1.7.) and (1.8.) become

$$a_3 c_3 + X_i \bar{d}_3 b_3 = 0 \quad (1.9.)$$

and

$$b_3 \bar{c}_3 + d_3 a_3 = 0. \quad (1.10.)$$

Since the algebra $A_{F_i}^{i-1} = F_i \otimes_{F_{i-1}} A_{i-1}$ is an algebra over F_{i-1} with basis $X^i \otimes f_j$, $i \in \mathbb{N}$ and $j = 1, 2, ...2^{i-1}$, we can write a_3, b_3, c_3, d_3 under the form $a_3 = \sum_{j \geq m} x_j X_i^j, b_3 = \sum_{j \geq n} y_j X_i^j, c_3 = \sum_{j \geq p} z_j X_i^j, d_3 = \sum_{j \geq r} w_j X_i^j$, where $x_j, y_j, z_j, w_j \in A_{i-1}, x_m, y_n, z_p, w_r \neq 0$. Since A_{i-1} is a division algebra, we have $x_m z_p \neq 0, w_r y_n \neq 0$, $y_n z_p \neq 0, w_r x_m \neq 0$. Using relations (1.9) and (1.10.), we have that $2m + p + r = 2n + p + r + 1$, which is false. Therefore,

the algebra A_i is a division algebra over the field $F_i = K(X_1, X_2, ..., X_i)$ of dimension 2^i.

3. An example of division algebra of dimension 2^t and prescribed level $2^k, t, k \in \mathbb{N}^*$

In his paper [O' Sh; 07], J. O'Shea gives a classification of the levels of composition algebras. For that, it is used only division quaternion and octonion algebras. Now, we try to extend some of these results to the algebras obtained by the Cayley-Dickson process.

Definition 3.1. The *level* of the algebra A, denoted by $s(A)$, is the least integer n such that -1 is a sum of n squares in A. The *sublevel* of the algebra A, denoted by $\underline{s}(A)$, is the least integer n such that 0 is a sum of $n+1$ nonzero squares of elements in A. If these numbers do not exist, then the level and sublevel are infinite.

Obviously, $\underline{s}(A) \leq s(A)$. We remark that, if in the Cayley-Dickson process, the quaternion algebra A_2 and the octonion algebra are split, then $s(A_2) = s(A_3) = 1$ [Pu, 05, Lemma 2.3.].

It is known that if the level of a field is finite, then its level is a power of 2.

Let A be an algebra over a field K obtained by the Cayley-Dickson process, of dimension $q = 2^t, T_C$ and T_P be its trace and pure trace forms.

Proposition 3.2. *If* $s(A) \leq n$ *then there is an element* $x \in A^n$ *such that* $(n \times T_C)(x) = -1.(-1$ *is represented by the quadratic form* $n \times T_C)$.

Proof. Let $y \in A, y = x_1 + x_2 f_2 + ... + x_q f_q$, $x_i \in K$, for all $i \in \{1, 2, ..., q\}$. Using the notations given in Introduction, we get $y^2 = x_1^2 + \beta_2 x_2^2 + ... + \beta_q x_q^2 + 2x_1 y''$, where $y'' = x_2 f_2 + ... + x_q f_q$. If -1 is a sum of n squares in A, then $-1 = y_1^2 + ... + y_n^2 = \left(x_{11}^2 + \beta_2 x_{12}^2 + ... + \beta_q x_{1q}^2 + 2x_{11} y_1''\right) + ... + \left(x_{n1}^2 + \beta_2 x_{n2}^2 + ... + \beta_q x_{nq}^2 + 2x_{n1} y_n''\right)$. Then we have

$-1 = \sum\limits_{i=1}^{n} x_{i1}^2 + \beta_2 \sum\limits_{i=1}^{n} x_{i2}^2 + ... + \beta_q \sum\limits_{i=1}^{n} x_{iq}^2$ and
$\sum\limits_{i=1}^{n} x_{i1}x_{i2} = \sum\limits_{i=1}^{n} x_{i1}x_{i3}... = = \sum\limits_{i=1}^{n} x_{i1}x_{in} = 0$, then $n \times T_C$ represents $-1.\Box$

In Proposition 3.2, we remark that the quadratic form $< 1 >\perp n \times T_C$ is isotropic.

Proposition 3.3. *For $n \in \mathbb{N}^*$, if the quadratic form $< 1 >\perp n \times T_P$ is isotropic over K, then $s(A) \leq n$.*

Proof. Case 1. If $-1 \in K^{*2}$, then $s(A) = 1$.
Case 2. $-1 \notin K^{*2}$. Since the quadratic form $< 1 >\perp n \times T_P$ is isotropic then it is universal. It results that $< 1 >\perp n \times T_P$ represent -1. Then, we have the elements $\alpha \in K$ and $p_i \in Skew(A)$, $i = 1, ..., n$, such that $-1 = \alpha^2 + \beta_2 \sum\limits_{i=1}^{n} p_{i2}^2 + ... + \beta_q \sum\limits_{i=1}^{n} p_{iq}^2$, and not all of them are zero.

i) If $\alpha = 0$, then $-1 = \beta_2 \sum\limits_{i=1}^{n} p_{i2}^2 + ... + \beta_q \sum\limits_{i=1}^{n} p_{iq}^2$. It results $-1 = \left(\beta_2 p_{12}^2 + ... + \beta_q p_{1q}^2\right) + ... + \left(\beta_2 p_{n2}^2 + ... + \beta_q p_{nq}^2\right)$. Denoting $u_i = p_{i2}f_2 + ... + p_{iq}f_q$, we have that $t(u_i) = 0$ and $u_i^2 = -n(u_i^2) = \beta_2 p_{i2}^2 + ... + \beta_q p_{iq}^2$, for all $i \in \{1, 2, ..., n\}$. We obtain $-1 = u_1^2 + ... + u_n^2$.

ii) If $\alpha \neq 0$, then $1 + \alpha^2 \neq 0$ and $0 = 1 + \alpha^2 + \beta_2 \sum\limits_{i=1}^{n} p_{i2}^2 + ... + \beta_q \sum\limits_{i=1}^{n} p_{iq}^2$. Multiplying this relation with $1 + \alpha^2$ and using Proposition 1.5., it results $0 = (1+\alpha^2)^2 + \beta_2 \sum\limits_{i=1}^{n} r_{i2}^2 + ... + \beta_q \sum\limits_{i=1}^{n} r_{iq}^2$. Therefore $-1 = \beta_2 \sum\limits_{i=1}^{n} r_{i2}'^2 + ... + \beta_q \sum\limits_{i=1}^{n} r_{iq}'^2$, where $r'_{ij} = r_{ij}(1 + \alpha)^{-1}, j \in \{2, 3, ..., q\}$ and we apply case i). Therefore $s(A) \leq n.\Box$

Lemma 3.4.[Sch; 85, p. 151] *Let $n = 2^k$, and $a_1, ...a_n, b_1, ..., b_n \in K$. Then there are elements $c_2, ..., c_n \in K$ such that*

$$(a_1^2 + ... + a_n^2)(b_1^2 + ... + b_n^2) = (a_1b_1 + ...a_nb_n)^2 + c_2^2 + ... + c_n^2.$$

Now we can state and prove some generalizations of J. O'Shea's results(Lemma 3.9, Proposition 3.2. and Proposition 3.3. from [O' Sh; 07]):u_{iu}

Proposition 3.5. *If $n \in \mathbb{N}^*$, $n = 2^k - 1$ such that $s(K) \geq 2^k$, then $s(A) \leq n$ if and only if $< 1 >\perp n \times T_P$ is isotropic.*

Proof. From Proposition 3.2, supposing that $s(A) \leq n$, we have $-1 = \sum\limits_{i=1}^{n} p_{i1}^2 + \beta_2 \sum\limits_{i=1}^{n} p_{i2}^2 + ... + \beta_q \sum\limits_{i=1}^{n} p_{iq}^2$ such that

$$\sum_{i=1}^{n} p_{i1}p_{i2} = \sum_{i=1}^{n} p_{i1}p_{i3} = ... = \sum_{i=1}^{n} p_{i1}p_{iq} = 0.$$

Since $s(K) \geq 2^k$, it results that $-1+\sum\limits_{i=1}^{n} p_{i1}^2 \neq 0$. Putting $p_{2^k1} = 1$ and $p_{2^k2} = p_{2^k3} = ...\ p_{2^kq} = 0$, we have

$$0 = \sum_{i=1}^{n+1} p_{i1}^2 + \beta_2 \sum_{i=1}^{n+1} p_{i2}^2 + ... + \beta_q \sum_{i=1}^{n+1} p_{iq}^2 \tag{3.1}$$

and $\sum\limits_{i=1}^{n+1} p_{i1}p_{i2} = \sum\limits_{i=1}^{n+1} p_{i1}p_{i3} = ... = \sum\limits_{i=1}^{n+1} p_{i1}p_{iq} = 0$. Multiplying (3.1.) by $\sum\limits_{i=1}^{n+1} p_{i1}^2$, since $\left(\sum\limits_{i=1}^{n+1} p_{i1}^2\right)^2$ is a square and using Lemma 3.4 for the products $\sum\limits_{i=1}^{n+1} p_{i2}^2 \sum\limits_{i=1}^{n+1} p_{i1}^2, ..., \sum\limits_{i=1}^{n+1} p_{iq}^2 \sum\limits_{i=1}^{n+1} p_{i1}^2$, we obtain

$$0 = \left(\sum_{i=1}^{n+1} p_{i1}^2\right)^2 + \beta_2 \sum_{i=1}^{n+1} r_{i2}^2 + ... + \beta_q \sum_{i=1}^{n+1} r_{iq}^2, \tag{3.2}$$

where $r_{i2}, ...r_{iq} \in K$, $n + 1 = 2^k$, $r_{12} = \sum\limits_{i=1}^{n+1} p_{i1}p_{i2} = 0$, $r_{13} = \sum\limits_{i=1}^{n+1} p_{i1}p_{i3} = 0, ..., r_{1q} = \sum\limits_{i=1}^{n+1} p_{i1}p_{iq} = 0$. Therefore, in the sums $\sum\limits_{i=1}^{n+1} r_{i2}^2$, ..., $\sum\limits_{i=1}^{n+1} r_{iq}^2$ we have n factors. From (3.2) , we get that $< 1 >\perp n \times T_P$ is isotropic.$\Box$

Proposition 3.6. *If* $s(K) \geq 2^k$, *then the quadratic form* $2^k \times T_C$ *is isotropic if and only if* $< 1 >\perp 2^k \times T_P$ *is isotropic.*

Proof. Since the form $< 1 >\perp 2^k \times T_P$ is a subform of the form $2^k \times T_C$, if the form $< 1 >\perp 2^k \times T_P$ is isotropic, we have that $2^k \times T_C$ is isotropic.

For the converse, supposing that $2^k \times T_C$ is isotropic, then we get

$$\sum_{i=1}^{2^k} p_i^2 + \beta_2 \sum_{i=1}^{2^k} p_{i2}^2 + ... + \beta_q \sum_{i=1}^{2^k} p_{iq}^2 = 0, \tag{3.3}$$

where $p_i, p_{ij} \in K, i = 1, ..., 2^k, \ j \in 2, ..., q$ and some of the elements p_i and p_{ij} are nonzero.

If $p_i = 0, \forall i = 1, ..., 2^k$, then $2^k \times T_P$ is isotropic, therefore $< 1 >\perp 2^k \times T_P$ is isotropic.

If $\sum_{i=1}^{2^k} p_i^2 \neq 0$, then, multiplying relation (3.3) with $\sum_{i=1}^{2^k} p_i^2$ and using Lemma 3.4 for the products $\sum_{i=1}^{2^k} p_{i2}^2 \sum_{i=1}^{2^k} p_i^2, ..., \sum_{i=1}^{2^k} p_{iq}^2 \sum_{i=1}^{2^k} p_i^2$, we obtain

$$(\sum_{i=1}^{2^k} p_i^2)^2 + \beta_2 \sum_{i=1}^{2^k} r_{i2}^2 + ... + \beta_q \sum_{i=1}^{2^k} r_{iq}^2 = 0,$$

then $< 1 >\perp 2^k \times T_P$ is isotropic.

Since $s(K) \geq 2^k$, the relation $\sum_{i=1}^{2^k} p_i^2 = 0$, for some $p_i \neq 0$, does not work. Indeed, supposing that $p_1 \neq 0$, we obtain $-1 = \sum_{i=2}^{2^k} (p_i p_1^{-1})^2$, false.$\square$

Proposition 3.7. *Let $n = 2^k - 1$ and $s(K) \geq 2^k$. Then $\underline{s}(A) \leq n$ if and only if $< 1 >\perp (n \times T_P)$ is isotropic or $(n+1) \times T_P$ is isotropic.*

Proof. Since $\underline{s}(A) \leq s(A)$, if $< 1 >\perp (n \times T_P)$ is isotropic, then, from Proposition 3.5, we have $\underline{s}(A) \leq n$. If $(n+1) \times T_P$ is isotropic, then there are the elements $p_{ij} \in K, i = 1, ..., 2^k, j \in 2, ..., q$, some of them are nonzero, such that $\beta_2 \sum_{i=1}^{2^k} p_{i2}^2 + ... + \beta_q \sum_{i=1}^{2^k} p_{iq}^2 = 0$. We obtain $0 = \left(\beta_2 p_{12}^2 + ... + \beta_q p_{1q}^2\right) + ... + \left(\beta_2 p_{n2}^2 + ... + \beta_q p_{nq}^2\right)$. Denoting $u_i = p_{i2} f_2 + ... + p_{iq} f_q$, we have $t(u_i) = 0$ and $u_i^2 = -n(u_i^2) = \beta_2 p_{i2}^2 + ... + \beta_q p_{iq}^2$, for all $i \in \{1, 2, ..., n\}$. Therefore $0 = u_1^2 + ... + u_n^2$. It results that $\underline{s}(A) \leq n$.

Conversely, if $\underline{s}(A) \leq n$, then there are the elements $y_1, ..., y_{n+1} \in A$, some of them must be nonzero, such that $0 = y_1^2 + ... + y_{n+1}^2$. As in the proof of Proposition 3.2., we obtain $0 = \sum_{i=1}^{n+1} x_{i1}^2 + \beta_2 \sum_{i=1}^{n+1} x_{i2}^2 + ... + \beta_q \sum_{i=1}^{n+1} x_{iq}^2$ and $\sum_{i=1}^{n+1} x_{i1} x_{i2} = \sum_{i=1}^{n+1} x_{i1} x_{i3} ... = \sum_{i=1}^{n+1} x_{i1} x_{in} = 0$. If all $x_{i1} = 0$, then $(n+1) \times T_P$ is isotropic. If $\sum_{i=1}^{n+1} x_{i1}^2 \neq 0$, then $(n+1) \times T_C$ is isotropic, or multiply-

ing the last relation with $\sum\limits_{i=1}^{2^k} x_{i1}^2$ and using Lemma 3.4 for the products $\sum\limits_{i=1}^{2^k} x_{i2}^2 \sum\limits_{i=1}^{2^k} x_{i1}^2, ..., \sum\limits_{i=1}^{2^k} x_{iq}^2 \sum\limits_{i=1}^{2^k} x_{i1}^2$, we obtain that $< 1 > \perp (n \times T_P)$ is isotropic. Since $s(K) \geq 2^k$, then the relation $\sum\limits_{i=1}^{n+1} x_{i1}^2 = 0$ for some $x_{i1} \neq 0$ is false. $\square$

Proposition 3.8. *If* $-1 \notin K^{*2}$, *then* $\underline{s}(A) = 1$ *if and only if either* T_C *or* $2 \times T_P$ *is isotropic.*

Proof. We apply Proposition 3.7 for $k = 1$.$\square$

Proposition 3.9. *Let* A *be an algebra obtained by the Cayley-Dickson process. The following statements are true:*

a) If -1 *is a square in* K, *then* $\underline{s}(A) = s(A) = 1$.

b) If $-1 \notin K^{*2}$, *then* $s(A) = 1$ *if and only if* T_C *is isotropic.*

Proof. a) If $-1 = a^2 \in K \subset A$, then $\underline{s}(A) = s(A) = 1$.

b) If $-1 \notin K^{*2}$ and $s(A) = 1$, then, there is an element $y \in A$ such that $-1 = y^2$, with $y = y_1 + y_2 f_2 + ... + y_q f_q$. Since $y^2 + 1 = 0$, then $y_1 = t(y) = 0$ and so $n(y) = 1$. Since $2T_C(y) = t(y^2) = -2n(y) = -2$, we obtain $T_C(y) = -1$, then

$$y^2 = -1 = \beta_2 y_2^2 + ... + \beta_q y_q^2,$$

therefore $0 = 1 + \beta_2 y_2^2 + ... + \beta_q y_q^2$. It results that T_C is isotropic.

Conversely, if T_C is isotropic, then there is $y \in A$, $y \neq 0$, such that $T_C(y) = 0 = y_1^2 + \beta_2 y_2^2 + ... + \beta_q y_q^2$. If $y_1 = 0$, then $T_C(y) = T_P(y) = 0$, so $y = 0$, false. If $y_1 \neq 0$, then $-1 = \left(\left(\frac{y_2}{y_1}\right) f_2 + ... + \left(\frac{y_q}{y_1}\right) f_q\right)^2$, obtaining $s(A) = 1$.$\square$

Remark 3.10. Using the above notations, if the algebra A is an algebra obtained by the Cayley-Dickson process, of dimension greater than 2 and if n_C is isotropic, then $s(A) = \underline{s}(A) = 1$. Indeed, if -1 is a square in K, the statement results from Proposition 3.9.a). If $-1 \notin K^{*2}$, since $n_C =< 1 > \perp -T_P$ and n_C is a Pfister form, we obtain that $-T_P$ is isotropic, therefore T_C is isotropic. Using Proposition 3.9., we have that $s(A) = \underline{s}(A) = 1$.

In the following, we give an example of division algebra of dimension 2^t and prescribed level 2^k.

Proposition 3.11. *Let* K *be a field such that* $s(K) = 2^k$, X *be an algebraically independent indeterminate over* K, D *be a finite-dimensional*

division $K-$algebra with involution $\varphi : A \to A, \varphi(x) = \bar{x}$ such that $s(D) = s(K)$, $D_1 = K(X) \otimes_K D$ and B be the $K(X)-$algebra obtained by application of the Cayley-Dickson process, with $\alpha = X$ to the $K(X)-$algebra D_1. Then B is a division algebra such that $s(B) = s(K)$.

Proof. We prove, first, that B is a division algebra. By straightforward calculations, using the same arguments like in the Brown's construction, given in Section 2, we obtain that B is a division algebra.(Here A_{i-1} is D, D_1 is $A_{F_i}^{i-1}, B$ is A_i.)

For the second part, since $s(B) \leq s(K) = n = 2^k$, we suppose that $s(B) \leq n-1$. It results that $-1 = y_1^2 + ... + y_{n-1}^2$, where $y_i \in B$, $y_i = a_{i1} + a_{i2}u$, $u^2 = X$, $a_{i1}, a_{i2} \in D_1$, some of y_i are nonzero. We have $y_i^2 = a_{i1}^2 + X\overline{a}_{i2}a_{i2} + (a_{i2}\overline{a}_{i1} + a_{i2}a_{i1})u, i \in \{1,2,...n-1\}$. It results $-1 = \sum\limits_{i=1}^{n-1} a_{i1}^2 + X \sum\limits_{i=1}^{n-1} \overline{a}_{i2}a_{i2}$, where $\psi = 1 \otimes \varphi$ is involution in $D_1, \psi(x) = \bar{x}$. We remark that $\overline{a}_{i2}a_{i2} \in K(X)$, $i \in \{1,..., n-1\}$. If $a_{i1} = \sum\limits_{j=1}^{m} \dfrac{p_{ji1}(X)}{q_{ji1}(X)} \otimes b_j$, with $b_j \in D$, $\dfrac{p_{ji1}(X)}{q_{ji1}(X)} \in K(X), i \in \{1,2,...,n-1\}$, $j \in \{1,2,...,m\}$.

$a_{i2} = \sum\limits_{j=1}^{m} \dfrac{r_{ji2}(X)}{w_{ji2}(X)} \otimes d_j$, with $d_j \in D$, $\dfrac{r_{ji2}(X)}{w_{ji2}(X)} \in K(X)$, $i \in \{1,2,...,n-1\}$, $j \in \{1,2,...,m\}$, it results

$$-1 = \sum_{i=1}^{n-1}(\sum_{j=1}^{m}\frac{p_{ji1}(X)}{q_{ji1}(X)} \otimes b_j)^2 + X\sum_{i=1}^{n-1}(\sum_{j=1}^{m}\frac{r_{ji2}(X)}{w_{ji2}(X)} \otimes d_j)(\sum_{j=1}^{m}\frac{r_{ji2}(X)}{w_{ji2}(X)} \otimes \bar{d}_j).$$

After clearing denominators, we obtain

$$-v^2(X) = \sum_{i=1}^{n-1}(\sum_{j=1}^{m}p'_{ji1}(X) \otimes b_j)^2 + X\sum_{i=1}^{n-1}(\sum_{j=1}^{m}r'_{ji2}(X) \otimes d_j)(\sum_{j=1}^{m}r'_{ji2} \otimes \overline{d}_j), \quad (3.4.)$$

where $v(X) = lcm\{q_{ji1}(X), w_{ji2}(X)\}, i \in \{1,2,...,n-1\}, j \in \{1,2,...,m\}$ and $p'_{ji1}(X) = v(X)p_{ji1}(X), r'_{ji2}(X) = v(X)r_{ji2}(X)$, $i \in \{1,..., n-1\}, j \in \{1,2,...,m\}$. We can write

$$v(X) = v_q X^q + v_{q+1}X^{q+1} + ..., v_q \in K, v_q \neq 0, \quad (3.5.)$$

$$\sum_{j=1}^{m} p'_{ji1}(X) \otimes b_j = \alpha_{r_i} X^{r_i} + \alpha_{r_i+1} X^{r_i+1} + \ldots.., \ \alpha_{r_i}, \alpha_{r_i+1}, \ldots. \in D, \alpha_{r_i} \neq 0, \tag{3.6.}$$

$$\sum_{j=1}^{m} r'_{ji2}(X) \otimes d_j = \beta_{s_i} X^{s_i} + \beta_{s_i+1} X^{s_i+1} + \ldots.., \beta_{s_i}, \beta_{s_i+1}, \ldots \in D, \beta_{s_i} \neq 0, \tag{3.7.}$$

$$\sum_{j=1}^{m} r'_{ji2} \otimes \overline{d}_j = \overline{\beta}_{s_i} X^{s_i} + \overline{\beta}_{s_i+1} X^{s_i+1} + \ldots.., \overline{\beta}_{s_i}, \overline{\beta}_{s_i+1}, \ldots \in D, \overline{\beta}_{s_i} \neq 0. \tag{3.8.}$$

By (3.4.), if $s = \min\limits_{i=\overline{1,n-1}} s_i$, $r = \min\limits_{i=\overline{1,n-1}} r_i$, in the left side the minimum degree is $2q$ (q possible zero) in the right side, the first sum has the minimum degree $2r$ (r possible zero) and the second term has the minimum degree $2s+1$. It results $q = r$ and $2r < 2s+1$. Replacing the relations (3.5.), (3.6.), (3.7.), (3.8.) in the relation (3.4.), if $r > 0$, we divide relation (3.4.) by X^{2r}, such that, in the new obtained relation the minimum degree in the both sides is zero. Putting $X = 0$ in this new relation, we have

$$-v_q^2 = \sum_{i=1}^{n-1} \alpha_{r_i}^2, \ \ \alpha_{r_i} \in D.$$

We obtain

$$-1 = \sum_{i=1}^{n-1} \left(\frac{\alpha_{r_i}}{v_q}\right)^2.$$

It results that $s(D) \leq n-1$, false.

Remark 3.13. Using Example 4.2. from [O' Sh; 07], we have that, if K_0 is a formally real field, then the field $F_0 = K_0(\left(2^k+1\right) \times < 1 >)$ has the level 2^k. If $D = A_0 = F_0, K = F_0, D_1 = K(X_1) \otimes_K A_0$, from Brown's construction and Proposition 3.11., the $K(X_1)-$algebra B, obtained by application of the Cayley-Dickson process with $\alpha = X_1$ to the $K(X_1)-$algebra D_1, is a division algebra of dimension 2 and level 2^k.

By induction, supposing that $D = A_{t-1}$ is a division algebra of dimension 2^{t-1} and level 2^k over the field $K = F_0(X_1, \ldots, X_{t-1})$, then, if $D = A_{t-1}$, $D_1 = K(X_t) \otimes_K A_{t-1}$ and B is the $K(X_t)-$algebra obtained by application of the Cayley-Dickson process with $\alpha = X_t$ to the $K(X_t)-$algebra D_1, then B is a division algebra of dimension 2^t and level 2^k.

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